FLUID VELOCITY AND PRESSURE DISTRIBUTIONS ALONG A PIPE WITH HOLES

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Fluid velocity and pressure distributions, averaged over the cross section, are found for a finite straight pipe containing holes or a slit of constant area per unit length.

1. We discuss, in the hydraulic approximation, the stationary flow of a fluid in a straight pipe with small closely spaced holes distributed quasicontinuously in the longitudinal direction, or a narrow slit or slits having a constant total area per unit length of pipe. If the pipe is closed at one end and is either horizontal or surrounded by fluid of the same density ρ, and if the quadratic law of resistance holds, the equations for the longitudinal motion of the fluid have the form [1, 2]

$$\left(\varkappa v^2 + \frac{p}{\rho}\right)' = -\frac{\lambda}{2} v^2, \quad v' = -\frac{\sigma}{L}\sqrt{\frac{2}{\rho} p}, \tag{1}$$

and if the pressure is specified at the pipe inlet the boundary conditions are

$$p(0) = p_0, \quad v(L) = 0. \tag{2}$$

A similar problem involving a single hole or several holes and branch pipes has been discussed [1-5] and the phenomenon of pressure reduction analyzed.

It is natural to write Eqs. (1) and (2) in dimensionless form:

$$(2\varkappa u^2 + \varphi)' = -\psi u^2, \quad u' = -\sigma\sqrt{\varphi}; \tag{3}$$

$$\varphi(0) = 1, \quad u(1) = 0. \tag{4}$$

Thus the process under consideration is characterized by two controlling dimensionless quantities σ and ψ, and the conditions for its simulation are σ = idem and ψ = idem [6].

It follows from (3) that u is always a monotonically decreasing function of x. We note that in the limiting case of a very long pipe when the second boundary condition is approximately satisfied, Eqs. (3) and (4) have a solution which is nearly exponential:

$$u = u_0 \exp\left(-\frac{\sigma}{u_0} x\right), \quad \varphi = \exp\left(-\frac{\sigma}{u_0} x\right), \tag{5}$$

where u_0 is determined from

$$(2\sigma)^{-1}\psi u_0^3 - 2\varkappa u_0^2 + 1 = 0. \tag{6}$$

By changing the independent variables

$$u = \frac{y}{2\varkappa}, \quad \varphi = z^2 \tag{7}$$

Eq. (3) is transformed into a simpler system (8) whose analytic solution is found and analyzed below.

The solutions of (3) and (4) were calculated on a Minsk-22 computer for a series of values of σ and a constant value of $\sigma/\psi = 0.256$ (Fig. 1, a and b), i.e. for a variable pipe length and linear characteristics independent of it.

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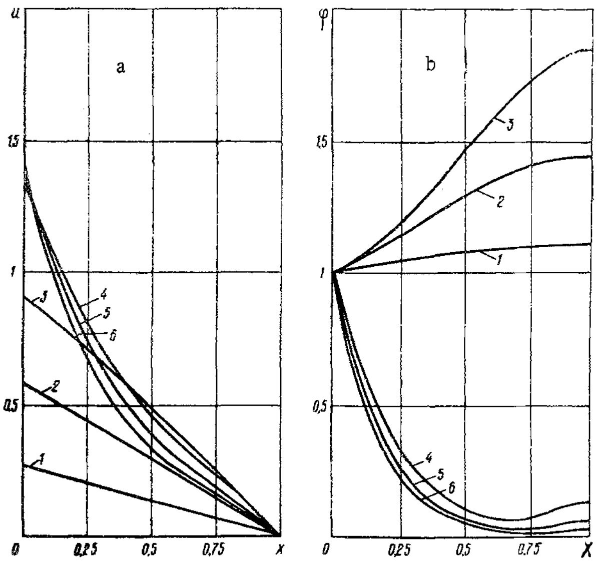

Fig. 1. Dimensionless velocity (a) and dimensionless gage pressure (b) as functions of the dimensionless coordinate x measured along the pipe. 1) $\sigma = 0.25$; 2) 0.5; 3) 0.75; 4) 3; 5) 3.5; 6) 4.

The solutions behave differently for $\sigma < 1$ and $\sigma > 1$. For $\sigma < 1$ u is nearly a linear function of x and $u_0(\sigma)$ increases rapidly. The pressure recovery predominates everywhere over the resistance, and the functions $\varphi(x)$ and $\varphi_1(\sigma)$ increase monotonically. For $\sigma > 1$ $u_0(\sigma)$ increases slowly, approaching the limit determined by (6), and the main part of the curve for u(x) falls with increasing σ. The function $\varphi(x)$ has a minimum, with the amount of the increase and $\varphi_1(\sigma)$ decreasing as σ increases. Both u(x) and $\varphi(x)$ approach exponentials (5).

The functions $u_0(\sigma)$ and $\varphi_1(\sigma)$ are shown in Fig. 2.

The theoretical results were tested experimentally. Two cylindrical pipes of length L = 2.25 m with inside diameters of 40 and 100 mm were investigated. Holes 4 mm in diameter were spaced along the generators with a distance of 25 mm between centers. The working medium was natural gas with an inlet gage pressure $p_0 = 175$ and 100 N/m^2. The values of p were measured with a TNZh-25 gage. The experimental conditions did not correspond exactly to the idealized case assumed in [1]. Nevertheless when the device permitted a measurement of the pressure drop the values of φ did not differ from the theoretical predictions by more than 2.6%.

2. We consider the system of differential equations

$$yy' + zz' = \alpha y^2, \quad y' = \beta z \, (\alpha\beta \neq 0) \tag{8}$$

or the equivalent equations: for y

$$y'y'' + \beta^2 y (y' - \alpha y) = 0 \tag{9}$$

and for z

$$[\gamma (zz')' + zz']^2 - z (z' - 2\alpha z)^2 (2\gamma z' + z) = 0 . \tag{10}$$

It should be noted, however, that the solutions of Eqs. (9) and (10) contain the solutions of (8) for both signs of β.

Making the substitutions

$$\xi^2 = y^2 + z^2 \;\; (\xi \geqslant 0), \quad \eta = \frac{y}{z}, \tag{11}$$

we transform (8) into

$$\xi' = \alpha \frac{\eta^2}{\eta^2 + 1} \xi, \quad \eta' = -\alpha [\eta^3 - \varepsilon (\eta^2 + 1)] . \tag{12}$$

The second of Eqs. (12) can be integrated directly by separation of variables to give

$$x = C_1 + \frac{1}{\alpha\eta_* (3\eta_* - 2\varepsilon)} \left[\ln \frac{\sqrt{\eta^2 + (\eta_* - \varepsilon)(\eta + \eta_*)}}{|\eta - \eta_*|} + \frac{3\eta_* - \varepsilon}{\sqrt{(\eta_* - \varepsilon)(3\eta_* + \varepsilon)}} \operatorname{arctg} \frac{2\eta + \eta_* - \varepsilon}{\sqrt{(\eta_* - \varepsilon)(3\eta_* + \varepsilon)}} \right], \tag{13}$$

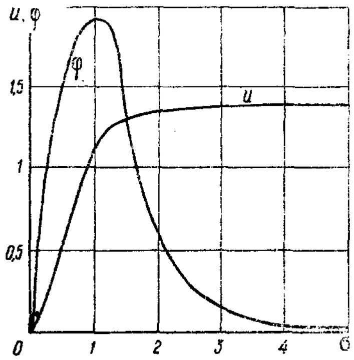

Fig. 2. u_0 and φ_1 as functions of σ.

which explicitly determines the function which is the inverse of $\eta = \eta(x)$. Here $\eta_* = \eta_*(\varepsilon)(\operatorname{sgn}\eta_* = \operatorname{sgn}\varepsilon,\ |\eta_*| > \max\{|\varepsilon|, |\varepsilon|^{1/3}\})$ is the only real root of $\eta^3 - \varepsilon(\eta^2 + 1) = 0$. $\eta(x)$ is an odd function of β.

From (12) ξ is found as the integral

$$\xi = \exp\left\{\int\limits_{\eta(x)} \frac{\eta^2}{(\eta^2+1)[\eta^3-\varepsilon(\eta^2+1)]}\,d\eta\right\}. \tag{14}$$

By evaluating the last integral we express $y = \xi\eta/\sqrt{\eta^2+1}$ and $z = \xi/\sqrt{\eta^2+1}$ in terms of η:

$$z = C_2\left\{|\eta_*-\eta|^{\eta_*-\varepsilon}[\eta^2+(\eta_*-\varepsilon)(\eta+\eta_*)]^{\eta_*-\frac{\varepsilon}{2}}\right\}^{-\frac{1}{3\eta_*-2\varepsilon}} \times \exp\left[\frac{\varepsilon}{3\eta_*-2\varepsilon}\sqrt{\frac{\eta_*-\varepsilon}{3\eta_*+\varepsilon}}\operatorname{arctg}\frac{2\eta+\eta_*-\varepsilon}{\sqrt{(\eta_*-\varepsilon)(3\eta_*+\varepsilon)}}\right] = \frac{y}{\eta}. \tag{15}$$

The system of transcendental equations (13) and (15) is the general solution of (8), and (13) and the second or first of Eqs. (15) are respectively the general solution of Eqs. (9) and (10). The solutions are written in parametric form by means of the parameter η. They are not contained in handbook [7]. Equations of the form (8)-(10) must be integrated numerically in specific cases.

3. We note particular solutions of (8)-(10) of the exponential type

$$\frac{y}{\eta_*} = z = C\exp\left(\frac{\beta}{\eta_*}x\right), \tag{16}$$

vanish at infinity $x = \pm\infty$ respectively for $\alpha \lessgtr 0$.

We assume, in accordance with our problems, that $\alpha < 0$ and $\beta < 0$, the argument is positive and has bounded variation $(0 \le x \le 1)$, and the boundary conditions are of the type

$$z(0) = 1,\ y(1) = 0 \text{ for } (8);\ y'(0) = \beta,\ y(1) = 0 \text{ for } (9);\quad z(0) = 1,\ z'(1) = 0 \text{ for } (10). \tag{17}$$

The corresponding particular solution has the form

$$x = 1 - \frac{1}{|\alpha|\eta_*(3\eta_*-2\varepsilon)}\left\{\frac{1}{2}\ln\left[\frac{\eta_*}{\eta_*-\varepsilon}\cdot\frac{\eta^2+(\eta_*-\varepsilon)(\eta+\eta_*)}{(\eta-\eta_*)^2}\right] + \frac{3\eta_*-\varepsilon}{\sqrt{(\eta_*-\varepsilon)(3\eta_*+\varepsilon)}}\operatorname{arctg}\left(\frac{\eta}{\eta+2\eta_*}\sqrt{\frac{3\eta_*+\varepsilon}{\eta_*-\varepsilon}}\right)\right\}, \tag{18}$$

$$z = \frac{y}{\eta} = \left\{\left|\frac{\eta_*-\eta_0}{\eta_*-\eta}\right|^{\eta_*-\varepsilon}\left[\frac{\eta_0^2+(\eta_*-\varepsilon)(\eta_*+\eta_0)}{\eta^2+(\eta_*-\varepsilon)(\eta+\eta_*)}\right]^{\eta_*-\frac{\varepsilon}{2}}\right\}^{\frac{1}{3\eta_*-2\varepsilon}}\exp\left\{\frac{\varepsilon}{3\eta_*-2\varepsilon}\sqrt{\frac{\eta_*-\varepsilon}{3\eta_*+\varepsilon}} \times \operatorname{arctg}\left[\sqrt{(\eta_*-\varepsilon)(3\eta_*+\varepsilon)}\frac{\eta-\eta_0}{(\eta_*+2\eta_0-\varepsilon)\eta+(\eta_*-\varepsilon)(2\eta_*+\eta_0)}\right]\right\}, \tag{19}$$

where $\eta_0 = \eta(0)$ is the root of the equation

$$\frac{1}{2}\ln\left[\frac{\eta_*}{\eta_*-\varepsilon}\cdot\frac{\eta_0^2+(\eta_*-\varepsilon)(\eta_0+\eta_*)}{\eta_0-\eta_*)^2}\right] + \frac{3\eta_*-\varepsilon}{\sqrt{(\eta_*-\varepsilon)(3\eta_*+\varepsilon)}}\operatorname{arctg}\left(\frac{\eta_0}{\eta_0+2\eta_*}\sqrt{\frac{3\eta_*+\varepsilon}{\eta_*-\varepsilon}}\right) = |\alpha|\eta_*(3\eta_*-2\varepsilon). \tag{20}$$

In accordance with (18)-(20) y(x) is a monotonically decreasing function. z(x) always has a maximum at x = 1 and increases monotonically for $\varepsilon \le \varepsilon_0$, where the function $\varepsilon_0 = \varepsilon_0(|\alpha|)$ is given by

$$\frac{3}{2}\ln\frac{\eta_*}{\eta_*-\varepsilon} + \frac{3\eta_*-\varepsilon}{\sqrt{(\eta_*-\varepsilon)(3\eta_*+\varepsilon)}}\operatorname{arctg}\left(\frac{\varepsilon}{2\eta_*+\varepsilon}\sqrt{\frac{3\eta_*+\varepsilon}{\eta_*-\varepsilon}}\right) = |\alpha|\,\eta_*(3\eta_*-2\varepsilon). \tag{21}$$

$\varepsilon > \varepsilon_0$ z(x) has a minimum. The magnitude of this minimum and the difference between the corresponding value of the argument

$$x = 1 - \frac{1}{|\alpha|\eta_*(3\eta_*-2\varepsilon)}\left[\frac{3}{2}\ln\frac{\eta_*}{\eta_*-\varepsilon} + \frac{3\eta_*-\varepsilon}{\sqrt{(\eta_*-\varepsilon)(3\eta_*+\varepsilon)}}\operatorname{arctg}\left(\frac{\varepsilon}{2\eta_*+\varepsilon}\sqrt{\frac{3\eta_*+\varepsilon}{\eta_*-\varepsilon}}\right)\right] \tag{22}$$

and unity decrease with increasing $|\alpha|$ for fixed ε.

We note that the particular solution (16) with C = 1 also satisfies the boundary conditions (17) at x = 0; the boundary conditions at x = 1 are only approximately satisfied for $|\alpha| > 1$, but more accurately the larger $|\alpha|$. In this sense the simple analytic solution (16) is limiting with respect to (18) and (19).

NOTATION

ρ	is the density of the fluid in the pipe;
$\varkappa$	is the Coriolis coefficient;
ψ, λ	are respectively the total and reduced resistance coefficients of the pipe;
σ	is the ratio of the total area of the holes or slits in the wall of the pipe to its inside cross-sectional area;
$l(0 \leq l \leq L)$	is the coordinate measured along the pipe;
$v = v(l)$	is the longitudinal velocity of the fluid averaged over the cross section of the pipe;
$p = p(l)$	is the gage pressure of the fluid averaged over the cross section of the pipe, $p_0 = p(0)$;
$x = l/L(0 \leq x \leq 1)$, $u = u(x) = v\sqrt{\rho/2p_0}$, $\varphi = \varphi(x) = p/p_0$	are dimensionless variables corresponding to l, v, and p; $u_0 = u(0)$, $\varphi_1 = \varphi(1)$;
y, z, ξ, η	are unknown functions of x;
α, $\beta(\alpha = -\psi/2\varkappa$, $\beta = -\sigma\sqrt{2\varkappa})$; $\gamma = 2\alpha/\beta^2$; $\varepsilon = \beta/\alpha$	are constants;
C, $C_i(i = 1, 2)$	are integration constants.

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